Intersection theorems for finite sets

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Robin Thomas 50

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Theorem (Frankl-Rödl (1987), \$250 problem of Erdős)

Suppose that $\mathcal{A} \subset 2^{[n]}$ and $|A \cap B| \neq n/4$ for all $A, B, \in \mathcal{A}$, and $n > n_0$. Then

$$|\mathcal{A}| < (1.99)^n.$$

Coding theory

- Q is an alphabet
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Suppose that p is prime and $d(\mathcal{C})$ is covered by t nonzero residue classes mod p. Then

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Theorem (Frankl-Rödl (1987))

Let $0 < \delta < 1/2$ and $\delta n < d < (1-\delta)n$, and d is even if $q = 2$. If $d \notin d(\mathcal{C})$, then $|\mathcal{C}| < (q-\varepsilon)^n$, where $\varepsilon = \varepsilon(\delta, q) > 0$.

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If true, then sharp by letting S be the vertices of a regular simplex, for example,

$$S = \{e_1, \ldots, e_d, v\}$$

where e_i is the unit vector with 1 in position i, and

$$v = \frac{1 - \sqrt{n+1}}{n}(1, \ldots, 1).$$

Results

- Borsuk (1932) $d = 2$
- Eggleston (1955) $d = 3$
- Hadwiger (1946) for all d if S is smooth and convex
- Riesling (1971) for all d if S is centrally symmetric
- Dekster (1995) for all d if S is a body of revolution
- Schramm (1988) number of pieces is at most $(\sqrt{3/2} + \epsilon)^d$, for all $\epsilon > 0$ and $d > d(\epsilon)$.

Counterexamples

Theorem (Kahn-Kalai (1993))

For large d, there exists a bounded $S \subset R^d$ such that every partition of S into pieces of smaller diameter has at least $(1.2)^{\sqrt{d}}$ parts. In particular, Borsuk's conjecture fails for $d = 1325$ and each $d > 2014$.

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Conjecture

There exists $c > 1$ such that for all d, there exists a bounded $S \subset R^d$ such that every partition of S into pieces of smaller diameter has at least c^d parts.

More Geometry

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Theorem (Frankl-Rödl (1987))

Given $r \geq 2$ and $n = d/4 \geq r$, there exists $\varepsilon = \varepsilon(r) > 0$ such that every set of more than $(2-\varepsilon)^d$ ± 1 vectors in R^d contains r pairwise orthogonal vectors.

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Conjecture (Erdős-Szemerédi (1978))

For every $\varepsilon > 0$, there is $n_0 = n_0(\varepsilon)$ such that if $n > n_0$ and $\mathcal{A} \subset 2^{[n]}$ with $|\mathcal{A}| > (2-\varepsilon)^n$, then $\mathcal{A}$ contains a weak delta system of size 3.

Theorem (Frankl-Rödl (1987))

Fix $r \geq 3$. Then there are $\eta = \eta(r)$ and $\varepsilon = \varepsilon(r)$ such that if $t = (1/4 \pm \eta)n$ and $\mathcal{A} \subset 2^{[n]}$ with $|\mathcal{A}| > (2-\varepsilon)^n$, then there are $A_1, \ldots, A_r \in \mathcal{A}$ with

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Recent work of Alon-Shpilka-Umans gives connections between this conjecture and algorithms for Matrix multiplication

Computer Science Applications

- Communication Complexity (Sgall 1999)
- Quantum Computing (Buhrman-Cleve-Wigderson 1998)
- Semidefinite Programming (Goemans-Kleinberg 1998, Hatami-Magen-Markakis 2009)

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Frankl-Rödl show it is about $(t/n)^2/2$.

The optimal ε_0

The binary entropy function is

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$$|\mathcal{A}| \leq \binom{n}{(n+t)/2} 2^{o(n)} = 2^{H\left(\frac{1}{2}+\frac{t}{2n}\right)n+o(n)}.$$

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For $n/3 < t < (1/2-\eta)n$, the construction $\mathcal{A} = \binom{[n]}{t}$ is better, and we conjecture it is optimal.

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Let $0 < \varepsilon < 1/5$ be fixed, $n > n_0(\varepsilon)$, $\varepsilon n < t < n/5$ and $\mathcal{A} \subset 2^{[n]}$. Suppose that

$$|A \cap B| \notin (t, t + n^{0.525})$$

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- The constant 0.525 is a consequence of the result of Baker-Harman-Pintz that there is a prime in every interval $(s - s^{0.525}, s)$ as long as s is sufficiently large.
- If we assume the Riemann Hypothesis, then 0.525 could be improved to $1/2 + o(1)$ using a result of Cramér.

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Eventown Theorem

Suppose that $\mathcal{A} \subset 2^{[n]}$ such that

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A Proof

Proof of Eventown:

- To each $A \in \mathcal{A}$, associate its incidence vector $v_A = (v_1, \ldots, v_n)$ where

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  - S is totally isotropic (meaning $x \cdot y = 0$ for $x, y \in S$)
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- So $|\mathcal{A}| \leq |S| \leq 2^{\lfloor n/2 \rfloor} = (1.4142..)^n$

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The length $\ell(M)$ of a set M is the maximum number of consecutive integers contained in M.

$\ell(M) \leq \ell$ if and only if $\overline{M}$ is $(\ell + 1)$-syndetic.

Bounds for small $\ell(M)$

Theorem (M-Rödl)

Let $M \subset [n]$ with $\ell(M) = \ell$. Suppose that $\mathcal{A} \subset 2^{[n]}$ is an M-intersecting family. Then

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- For example, if $[n] \setminus M = \{0, n/10^4, 2n/10^4, \ldots, \}$, then
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- The 1.622 is probably not sharp, just a result of the proof

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$$|\mathcal{A}| < 2^{n/2+o(n)}.$$
- This is the first non-linear-algebraic proof of an asymptotic version of the Eventown Theorem; it applies in more general scenarios though doesn't give bounds as precise as $2^{n/2}$.

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Theorem (M-Rödl)

Let $M \subset [n]$ with $\ell(M) = \ell$. Suppose that $(\mathcal{A}, \mathcal{B})$ is an M-intersecting pair of families in $2^{[n]}$. Then

$$|\mathcal{A}||\mathcal{B}| < \min\left\{2.631^n \times 10^{4\ell+10}, \quad 2^{n+2\ell \log^2 n}\right\}.$$

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(A4) if $h(L), h(L') \leq s$, then either

$$h(L' \cap L) \leq s - 1 \qquad \text{or} \qquad h(L' \cap (L - 1)) \leq s - 1.$$

Sgall's theorem

Theorem (Sgall (1999))

Suppose that $(\mathcal{A}, \mathcal{B})$ is an M-intersecting pair of families in $2^{[n]}$ and $h(M) \leq s \leq n+1$. Then

$$|\mathcal{A}||\mathcal{B}| \leq 2^{n+s-1}\binom{n}{s-1}.$$

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Theorem (M-Rödl)

There exists a height function h such that for every $M \subset [n]$,

$$h(M) \leq 1 + 2\ell(M)\log n.$$

Applying this bound in Sgall's Theorem yields $|\mathcal{A}||\mathcal{B}| < 2^{n+2\ell\log^2 n}$.

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Happy Birthday Robin!